Borel sets which are null or non-σ-finite for every translation invariant measure

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Abstract

We show that the set of Liouville numbers is either null or non-σ-finite with respect to every translation invariant Borel measure on $\mathbb{R}$, in particular, with respect to every Hausdorff measure $\mathcal{H}^g$ with gauge function g. This answers a question of D. Mauldin. We also show that some other simply defined Borel sets like non-normal or some Besicovitch-Eggleston numbers, as well as all Borel subgroups of $\mathbb{R}$ that are not F_σ possess the above property. We prove that, apart from some trivial cases, the Borel class, Hausdorff or packing dimension of a Borel set with no such measure on it can be arbitrary.

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Introduction

In many branches of mathematics a standard tool is that 'nicely defined' sets admit natural probability measures. For example, limit sets in the theory of Iterated Function Systems or Conformal Dynamics as well as self-similar sets in Geometric Measure Theory are usually naturally equipped with an invariant Borel measure, very often with a Hausdorff or packing measure. In many situations the sets in consideration are unbounded, for example periodic, so we cannot hope for an invariant probability measure. Similarly, the trajectories of the Brownian motion are of positive σ-finite $\mathcal{H}^g$-measure with probability 1, where the gauge function g is $g(t) = t^2 \log\log \frac{1}{t}$ in case of planar Brownian motion and $g(t) = t^2 \log \frac{1}{t} \log\log\log \frac{1}{t}$ in dimension 3 and higher. Therefore the natural notion to work with is that of an invariant positive and σ-finite measure.

It is natural to ask if there is some sort of unified theory behind the existence of these measures, for example, one is tempted to ask if every Borel subset of $\mathbb{R}^n$ of some 'regular structure' is positive and σ-finite for some Hausdorff measure, or at least admits a positive and σ-finite invariant measure. In particular, R. D. Mauldin ([13], [4] and see [3] and [2] for partial and related results) formulated this question about a specific well-known set of very nice structure; the set of Liouville numbers, denoted by L:

Definition 0.1

$$L = \left\{ x \in \mathbb{R} \setminus \mathbb{Q} : \forall n \in \mathbb{N} \; \exists p, q \in \mathbb{N} \; (q \geq 2) \text{ such that } \left| x - \frac{p}{q} \right| < \frac{1}{q^n} \right\}.$$

Question 0.2 *(Mauldin) Is there a translation invariant Borel measure on $\mathbb{R}$ such that the set of Liouville numbers is of positive and σ-finite measure?*

Note that we of course do not require that the measure be σ-finite on $\mathbb{R}$. Not only because Hausdorff measures are non-σ-finite on $\mathbb{R}$, but also because it is well-known that every σ-finite translation invariant Borel measure on the real line is a constant multiple of Lebesgue measure.

As we will answer this question in the negative, we introduce a definition.

Definition 0.3 A nonempty Borel set $B \subset \mathbb{R}$ is said to be *immeasurable* if it is either null or non-σ-finite for every translation invariant Borel measure on $\mathbb{R}$.

The main result of this paper will be Theorem 1.1 stating that the set of Liouville numbers is immeasurable. Then in the second part of the paper we show using various methods that there are other well-known 'nice' immeasurable sets. Specifically, we show that the set of non-normal numbers, the complement of the set of Besicovitch-Eggleston numbers, $BE(1, 0)$ (one of the Besicovitch-Eggleston classes itself) are all immeasurable. One of the main tool is Theorem 2.10 stating that every Borel but not F_σ additive subgroup of $\mathbb{R}$ is immeasurable. Using this we also show that there are immeasurable sets of arbitrary Borel

class (except of course open, as sets of positive Lebesgue measure are obviously not immeasurable). Similarly, we provide examples of immeasurable sets of arbitrary Hausdorff or packing dimension.

We note here that it is not only the regular structure of the sets considered here that makes it difficult to prove immeasurability. Even it is highly nontrivial to construct some immeasurable set. The two papers [10] and [5] containing the two known examples are entirely devoted to the constructions of the two immeasurable sets.

We would like to point out that in this paper a *Borel measure* is a measure defined on a σ-algebra *containing* the Borel sets. A *group* is always an additive subgroup of $\mathbb{R}$. Otherwise the notation we follow throughout the paper can be found for example in [11] or [8] and [9]. λ denotes Lebesgue measure, $\mathrm{int}(A)$ is the interior of the set A, $A+B = \{a+b : a \in A, b \in B\}$, $A+t = \{a+t : a \in A\}$, $\mathrm{card}\,(X)$ is the cardinality of the set X, a G_δ set is the countable intersection of open sets, a $G_{\delta\sigma}$ set is the countable union of G_δ sets, etc. By *Cantor set* we mean a set homeomorphic to the classical 'middle-thirds' Cantor set.

1 Liouville numbers

In this section we answer the question of Mauldin.

Theorem 1.1 *The set of Liouville numbers is immeasurable; that is, either null or non-σ-finite for every translation invariant Borel measure on $\mathbb{R}$.*

It is well-known and easy to check that L is of Lebesgue measure zero, dense, G_δ and periodic mod $\mathbb{Q}$ (that is, $L+q = L$ for every $q \in \mathbb{Q}$). Hence the above theorem is a corollary to the following.

Theorem 1.2 *Let $B \subset \mathbb{R}$ be a nonempty G_δ set of Lebesgue measure zero. Assume that $\{t \in \mathbb{R} : B+t \subset B\}$ is dense in $\mathbb{R}$. Then B is immeasurable.*

The proof of Theorem 1.2 will be based on two lemmas. The first one is reminiscent of [16], where similar results are proved for finite measures using more complicated methods.

Lemma 1.3 *Let B be a Borel set of Lebesgue measure zero and μ a Borel measure on $\mathbb{R}$ for which B is positive and σ-finite. Then*

(i) $\mu(B \cap (B+t)) = 0$ *for λ-a.e. t,*

(ii) there exists a Borel set $B' \subset B$ with $\mu(B') > 0$ and $\mathrm{int}(B'-B') = \emptyset$,

(iii) there exists a compact set $C \subset B$ with $\mu(C) > 0$ and $\mathrm{int}(C-C) = \emptyset$

Lemma 1.4 *Let B be a dense G_δ set such that $\{t \in \mathbb{R} : B+t \subset B\}$ is dense in $\mathbb{R}$, and $C \subset B$ be a compact set with $\mathrm{int}(C-C) = \emptyset$. Then there are uncountably many (in fact, continuum many) disjoint translates of C inside B.*

It is easy to see that applying Lemma 1.4 to C of Lemma 1.3 *(iii)* yields Theorem 1.2

In the rest of the section we prove the two lemmas.

Proof. (Lemma 1.3) *(i)* Let $\mu : \mathcal{S} \to [0, \infty]$ be the given measure, where $\mathcal{S}$ is a σ-algebra of subsets of $\mathbb{R}$ containing all Borel sets. Define a new measure μ_B by

$$\mu_B(S) = \mu(B \cap S) \text{ for every } S \in \mathcal{S}.$$

μ_B is clearly a σ-finite Borel measure on $\mathbb{R}$. Define

$$\widetilde{B} = \{(x, y) \in \mathbb{R}^2 : x + y \in B\}.$$

This is clearly a Borel set, hence $\mu_B \times \lambda$-measurable. As both measures are σ-finite, we can apply the Fubini theorem to $\widetilde{B}$. Vertical sections of $\widetilde{B}$ are of the form

$$\{y \in \mathbb{R} : x + y \in B\} = \{y \in \mathbb{R} : y \in B - x\} = B - x,$$

therefore are all of Lebesgue measure zero. By Fubini, $[\mu_B \times \lambda](\widetilde{B}) = 0$, and so λ-a.e. horizontal section of $\widetilde{B}$ is of μ_B-measure zero. A horizontal section is $\{x \in \mathbb{R} : x + y \in B\} = B - y$, therefore for λ-a.e. y we obtain $0 = \mu_B(B - y) = \mu(B \cap (B - y))$. Replacing y by $-t$ yields the result.

(ii) By *(i)* we can choose a countable dense set $D \subset \mathbb{R}$ such that $\mu(B \cap (B + d)) = 0$ for every $d \in D$. Define

$$B' = B \setminus \bigcup_{d \in D} (B + d).$$

It is easy to check that $\mu(B') = \mu(B) > 0$ and $D \cap (B' - B') = \emptyset$, so the proof of *(ii)* is complete.

(iii) It is sufficient to find a compact set $C \subset B'$ of positive μ-measure. Since $B' \subset B$, B' is σ-finite for μ. Let $B' = \cup_{n=0}^{\infty} S_n$, where $S_n \in \mathcal{S}$ and $\mu(S_n) < \infty$. Since $\mu(B') > 0$, there exists $S = S_n \subset B'$ such that $0 < \mu(S) < \infty$. Define

$$\mu_S(A) = \mu(S \cap A) \text{ for every Borel set } A \subset \mathbb{R}.$$

Note that in contrast with the above μ_B, this time the measure is defined only for Borel sets.

μ_S is clearly a finite measure on the Borel sets, hence inner regular w.r.t. compact sets [9, Thm 17.11]. Apply this to B', a Borel set with $\mu_S(B') = \mu(S \cap B') = \mu(S) > 0$, and obtain a compact set $C \subset B'$ with $\mu_S(C) = \mu(S \cap C) > 0$. So $\mu(C) > 0$ follows, and the proof of Lemma 1.3 is complete. $\square$

Proof. (Lemma 1.4) Let

$$T = \{t \in \mathbb{R} : C + t \subset B\}.$$

B is G_δ, so there are open sets U_n such that $B = \cap_{n=0}^{\infty} U_n$. Clearly $C + t \subset \cap_{n=0}^{\infty} U_n$ iff $C + t \subset U_n$ holds for every $n \in \mathbb{N}$. Therefore $T = \cap_{n=0}^{\infty} G_n$, where

$G_n = \{t \in \mathbb{R} : C + t \subset U_n\}$. As C is compact and U_n is open, G_n is open. Note that G_n is also dense by our assumption.

It is clearly sufficient to construct a Cantor set $P \subset T$ with the property that $(C+p_0) \cap (C+p_1) = \emptyset$ holds for every pair of distinct points $p_0, p_1 \in P$. We define P via a usual Cantor scheme as follows. Let 2^n stand for the set of 0–1 sequences of length n. We define nondegenerate compact intervals I_s for every $n \in \mathbb{N}$ and $s \in 2^n$ by induction on n (we also make sure that at level n all intervals are of length at most $\frac{1}{n}$). Fix $I_\emptyset$ such that $I_\emptyset \subset G_0$. Once I_s is already defined for some $s \in 2^n$, we pick $x \in I_s \cap G_{n+1}$. As G_{n+1} is open dense and $C - C$ is nowhere dense we can find $y \in [I_s \cap G_{n+1}] \setminus [(C - C) + x]$. This ensures $(C + x) \cap (C + y) = \emptyset$, as otherwise $c_0 + x = c_1 + y$ for some $c_0, c_1 \in C$, so $y = (c_0 - c_1) + x$, a contradiction. By compactness we can find disjoint $I_{s^\frown 0}, I_{s^\frown 1} \subset I_s \cap G_{n+1}$ such that $x \in I_{s^\frown 0}$, $y \in I_{s^\frown 1}$ and $(C + I_{s^\frown 0}) \cap (C + I_{s^\frown 1}) = \emptyset$.

Now define

$$P = \bigcap_{n=0}^{\infty} \bigcup_{s \in 2^n} I_s.$$

Then clearly P is a Cantor set, $P \subset T = \cap_{n=0}^{\infty} G_n$ as $I_s \subset G_n$ for every n and $s \in 2^n$. Moreover, if $p_0, p_1 \in P$, $p_0 \neq p_1$ then there are n and $s \in 2^n$ such that $p_0 \in I_{s^\frown 0}$ and $p_1 \in I_{s^\frown 1}$ (or the other way around), but then $(C+p_0) \cap (C+p_1) = \emptyset$ holds since $(C + I_{s^\frown 0}) \cap (C + I_{s^\frown 1}) = \emptyset$. This completes the proof of Lemma 1.4. □

Remark 1.5 Theorem 1.2 holds (with essentially the same proof) in $\mathbb{R}^n$, more generally in locally compact abelian Polish groups as well. Of course, Lebesgue measure is replaced by Haar measure.

No assumptions of Theorem 1.2 can be omitted. The empty set is not immeasurable by definition. The example of $\mathbb{Q}$ and the counting measure shows that the G_δ assumption is essential. Sets of positive Lebesgue measure are clearly not immeasurable. The density of $\{t \in \mathbb{R} : B + t \subset B\}$ is also important, as the example $B = L \cup C_{1/3}$ shows, where $C_{1/3}$ is the classical middle-thirds Cantor set. Indeed, $\log 2/\log 3$-dimensional Hausdorff measure is positive and finite on B, since it is easy to see that $\dim_H L = 0$; that is, Hausdorff dimension of L is zero (see [11] for the definition).

Theorem 1.2 provides a lot of immeasurable sets in the following sense. Let A be a (nonempty) set of Lebesgue measure zero, and let B be a G_δ set of measure zero containing $A + \mathbb{Q}$. Then $\cap_{q \in \mathbb{Q}} (B + q)$ fulfills all conditions of Theorem 1.2, hence it is immeasurable. Clearly $A \subset \cap_{q \in \mathbb{Q}} (B + q)$, therefore every Lebesgue nullset can be covered by an immeasurable set. Combining this with Lemma 2.2 *(i)* we obtain that the σ-ideal generated by the immeasurable sets is the Lebesgue null ideal.

2 Non-normal numbers, classes of Besicovitch-Eggleston numbers and Borel groups

In this section we provide some more examples of well-known sets that are immeasurable.

The following definition is classical.

Definition 2.1 A real number $x \in [0,1)$ is called *normal* if its decimal expansion $x = 0.d_1 d_2 \ldots$ satisfies

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} d_i}{n} = 4.5.$$

A real number $x \in \mathbb{R}$ is called normal if its fractional part, $\{x\}$ is normal. Note that for negative numbers this is *not* the usual decimal expansion. The set of normal numbers is denoted by N.

It is well-known and easy to check using the Strong Law of Large Numbers that λ-a.e. real number is normal.

Our next goal is to prove that the set of non-normal numbers, $\mathbb{R} \setminus N$ is immeasurable. Note that the Hausdorff dimension of this set is 1 (see [1]), while we already mentioned that $\dim_H L = 0$. Moreover, $\mathbb{R} \setminus N$ is not G_δ, since its exact Borel class is $G_{\delta\sigma}$ (that is, $\mathbb{R} \setminus N \in G_{\delta\sigma} \setminus F_{\sigma\delta}$, see [9, Ex 23.7]).

Before the proof we need one more easy lemma.

Lemma 2.2 *(i) The class of immeasurable sets is closed under countable unions.*

(ii) If the symmetric difference $A \triangle B$ is countable then A is immeasurable iff B is immeasurable.

Proof. *(i)* This clearly follows from the definition.

(ii) If A and B are nonempty countable then counting measure works for both. Otherwise all measures we consider are continuous; that is, singletons are of measure zero. $\square$

Theorem 2.3 *The set $\mathbb{R} \setminus N$ of non-normal numbers is immeasurable; that is, either null or non-σ-finite for every translation invariant Borel measure on $\mathbb{R}$.*

Proof. By Lemma 2.2 *(ii)* it is enough to show that $(\mathbb{R}\setminus N)\setminus\mathbb{Q}$ is immeasurable. Write this set as

$$(\mathbb{R} \setminus N) \setminus \mathbb{Q} = \bigcup_{k=1}^{\infty} A_k \cup \bigcup_{k=1}^{\infty} B_k,$$

where $A_k = \left\{x \in \mathbb{R} \setminus \mathbb{Q} : \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} d_i}{n} \geq 4.5 + \frac{1}{k}\right\}$ and similarly $B_k = \left\{x \in \mathbb{R} \setminus \mathbb{Q} : \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} d_i}{n} \leq 4.5 - \frac{1}{k}\right\}$. By Lemma 2.2 (i) it is enough to

prove that the sets A_k and B_k are immeasurable for every k. The other case being similar we check this only for A_k.

We show that Theorem 1.2 applies. As a.e. number is normal, A_k is of Lebesgue measure zero. Moreover, it is easy to see that it is dense and periodic modulo a dense set, namely the set of real numbers with finitely many nonzero digits. Therefore what remains to be checked is that A_k is G_δ.

$$A_k = \bigcap_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left\{ x \in \mathbb{R} \setminus \mathbb{Q} : \frac{\sum_{i=1}^{n} d_i}{n} > 4.5 + \frac{1}{k} - \frac{1}{l} \right\},$$

and $\left\{ x \in \mathbb{R} \setminus \mathbb{Q} : \frac{\sum_{i=1}^{n} d_i}{n} > 4.5 + \frac{1}{k} - \frac{1}{l} \right\}$ is clearly an open subset relative to $\mathbb{R} \setminus \mathbb{Q}$, hence A_k is G_δ relative to $\mathbb{R} \setminus \mathbb{Q}$, hence A_k is G_δ in $\mathbb{R}$ as well. □

Our next example concerns the so called Besicovitch-Eggleston numbers. These are the numbers for which 'the asymptotic frequency of every digit exists': Let $\{x\} = 0.d_1 d_2 \dots$ be the expansion of the fractional part of the real number x in base $b \geq 2$. If there are two such expansions, we choose the one with finitely many nonzero digits.

Definition 2.4 Let $\alpha_0, \dots, \alpha_{b-1}$ be real numbers such that $\alpha_d \geq 0$ for every $d = 0, \dots, b-1$ and $\sum_{d=0}^{b-1} \alpha_d = 1$.

$$BE(\alpha_0, \dots, \alpha_{b-1}) =$$

$$\left\{ x \in \mathbb{R} : \lim_{n \to \infty} \frac{\operatorname{card}\left(\{i : 1 \leq i \leq n, d_i = d\}\right)}{n} = \alpha_d \text{ for every } d = 0, \dots, b-1 \right\}.$$

The union of all these classes (for a fixed positive integer b) is denoted by BE_b.

Theorem 2.5 *For every integer $b \geq 2$ the set $\mathbb{R} \setminus BE_b$ is immeasurable.*

Proof. The proof is very similar to that of Theorem 2.3. We can again restrict ourselves to the set of irrational numbers. Write

$$(\mathbb{R} \setminus BE_b) \setminus \mathbb{Q} = \bigcup_{d=0}^{b-1} \bigcup_{\substack{p<q \\ p,q \in \mathbb{Q} \cap [0,1]}} \left(C_q^d \cap D_p^d \right),$$

where

$$C_q^d = \left\{ x \in \mathbb{R} \setminus \mathbb{Q} : \limsup_{n \to \infty} \frac{\operatorname{card}\left(\{i : 1 \leq i \leq n, d_i = d\}\right)}{n} \geq q \right\}$$

and

$$D_p^d = \left\{ x \in \mathbb{R} \setminus \mathbb{Q} : \liminf_{n \to \infty} \frac{\operatorname{card}\left(\{i : 1 \leq i \leq n, d_i = d\}\right)}{n} \leq p \right\}.$$

It is again easy to see using the Strong Law of Large Numbers that Lebesgue a.e. number is in BE_b, namely in $BE(\frac{1}{b}, \dots, \frac{1}{b})$, hence every C_q^d and D_p^d is of

Lebesgue measure zero. Moreover, these sets are invariant under all translations from the dense set of numbers with finitely many nonzero digits. So it is enough to show that C_q^d and D_p^d are G_δ in $\mathbb{R} \setminus \mathbb{Q}$. But this is clear, as

$$C_q^d = \bigcap_{l=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left\{ x \in \mathbb{R} \setminus \mathbb{Q} : \frac{\operatorname{card}\left(\{i : 1 \le i \le n, d_i = d\}\right)}{n} > q - \frac{1}{l} \right\},$$

and similarly for D_p^d. $\square$

So far all immeasurable sets in this paper were comeager. Our next example is easily seen to be meager. Not surprisingly we need new methods to prove immeasurability here.

Definition 2.6 Let A be a set of natural numbers. The expression

$$\lim_{n \to \infty} \frac{\operatorname{card}\left(A \cap [1, n]\right)}{n}$$

(if exists) is called the *density of* A.

It is easy to see that sets of zero density form an ideal on the natural numbers. Dually, sets of density 1 form a filter.

Theorem 2.7 *$BE(1,0)$ is immeasurable; that is, the set of real numbers with the property that in the dyadic expansion the 1's form a set of zero density is either null or non-σ-finite for every translation invariant Borel measure on $\mathbb{R}$.*

This result will follow from the following more general statement.

Theorem 2.8 *Let B be a Borel set such that $B + B \subset B$. If $B - B$, the group generated by B, is not F_σ then B is immeasurable.*

Proof. Assume towards a contradiction that μ is a translation invariant measure for which B is positive and σ-finite. As in the proof of Lemma 1.3 *(iii)* we obtain that there exists a compact set $C \subset B$ of positive μ-measure. We get a contradiction by showing that C has uncountably many disjoint translates in B. We define the transfinite sequence of numbers $\{t_\alpha : \alpha < \omega_1\}$ by transfinite induction, so that the sets $\{C + t_\alpha : \alpha < \omega_1\}$ are pairwise disjoint. Clearly $(C + t_\alpha) \cap (C + t_\beta) = \emptyset$ iff $t_\alpha \notin C - C + t_\beta$. Throughout the induction we choose the numbers from B, which implies $C + t_\alpha \subset B$ by our assumption, so what remains to show is that the induction cannot get stuck at some countable ordinal. At step α our task is to find $t_\alpha \in B$ so that $t_\alpha \notin C - C + t_\beta$ for every $\beta < \alpha$. These latter sets are in $B - B$ because this is a group, and we claim that they cannot cover B. Indeed, if $B \subset \cup_{\beta<\alpha}(C - C + t_\beta)$ then $B - B = \cup_{\beta,\gamma<\alpha}[(C - C + t_\beta) - (C - C + t_\gamma)]$, so $B - B$ is F_σ, which is a contradiction. $\square$

Before we finish the proof of Theorem 2.7 we need one more lemma, which provides an example of an immeasurable group.

Lemma 2.9 *Denote*

$$G = \{x \in \mathbb{R} : d_n = d_{n+1} \text{ for all } n \text{ but a set of density zero}\},$$

where $\{x\} = 0.d_1d_2\ldots$ *is the dyadic expansion of the fractional part of* x*. Then*

(i) $BE(1,0) - BE(1,0) = G$,

(ii) G *is a Borel but not* F_σ *group.*

Proof. *(i)* Put $B = BE(1,0)$ and define $S(x) = \{n \in \mathbb{N} : d_n \neq d_{n+1}\}$, where $\{x\} = 0.d_1d_2\ldots$ is the dyadic expansion of the fractional part of x. Then $G = \{x \in \mathbb{R} : S(x) \text{ is of density zero}\}$. We have to show that $B - B = G$. Note that all dyadic rationals (numbers with only finitely many nonzero digits) are in B as we chose the expansion that is eventually zero.

First we check $B - B \subset G$. Fix $b_1, b_2 \in B$, and write $b_1 - b_2 = b_1 + (-b_2)$. If b_2 is a dyadic rational then so is $-b_2$, hence all but finitely many digits of $b_1 + (-b_2)$ and b_1 coincide, hence $b_1 - b_2 \in B$. So we can assume that b_2 is not a dyadic rational, and then we obtain the expansion of (the fractional part of) $-b_2$ by replacing 0's with 1's and vice versa. Let d_i, e_i and f_i be the digits of the expansions of b_1, $-b_2$ and $b_1 + (-b_2)$, respectively. Then the set $A = \{i : d_i = d_{i+1} = 0, e_i = e_{i+1} = 1\}$ is of density 1. If $i \in A$ then either $f_i = f_{i+1} = 0$ or $f_i = f_{i+1} = 1$ (depending on whether there is a carried digit from the $i + 2^{nd}$ place), hence $i \notin S(b_1 + (-b_2))$. Therefore $S(b_1 + (-b_2))$ is of density zero and so $b_1 + (-b_2) \in G$.

Now we show that $G \subset B - B$. Given $x \in G$ with expansion $\{x\} = 0.d_1d_2\ldots$ we construct $b \in B$ with expansion $\{b\} = 0.e_1e_2\ldots$ such that $x + b \in B$. We can assume that x is not a dyadic rational, otherwise $b = 0$ works. Define b as follows: let $e_i = 1$ iff $d_i = 1$ and $d_{i+1} = 0$. Since $x \in G$ we have that $S(x)$ is of density zero, therefore $b \in B$. Let $\{x + b\} = 0.f_1f_2\ldots$. It suffices to check that $f_i = 1$ iff $d_i = 0$ and $d_{i+1} = 1$, hence then $\{i : f_i = 1\}$ is a subset of $S(x)$, therefore is of density zero, thus $x + b \in B$. Suppose first that $d_{i+1} = 0$. Then $e_{i+1} = 0$, so there is no carried digit from this place. Moreover $e_i = d_i$, so $f_i = 0$. Suppose now that $d_{i+1} = 1$. Let k be the minimal integer such that $i + 2 \leq k$ and $d_k = 0$. Then $e_j = 0$ for $i \leq j \leq k-2$ and for $j = k$, and $e_{k-1} = 1$. Consequently, there is no carried digit from the k^{th} place, so $f_{k-1} = 0$ and there is a carried digit from the $k - 1^{st}$ place, and so on there is a carried digit for every $i + 1 \leq j \leq k - 1$, and at the end $f_i = 1$ iff $d_i = 0$.

(ii) $B = BE(1,0)$ and G are clearly Borel sets. As B is easily seen to be closed under addition, $G = B - B$ is a group. So we have to check that G is not F_σ.

Suppose towards a contradiction that it is. First we construct a continuous map $f : [0,1) \setminus \mathbb{D} \to [0,1) \setminus \mathbb{D}$ (where $\mathbb{D}$ is the set of dyadic rationals), with the property that $f^{-1}(G \cap ([0,1) \setminus \mathbb{D})) = B \cap ([0,1) \setminus \mathbb{D})$. For $x = 0.d_1d_2\ldots$ construct the digits of $f(x) = 0.e_1e_2\ldots$ as follows. Let $d_1 = e_1$, and then define e_n recursively such that $e_{n+1} = e_n$ iff $d_n = 0$. It is easy to check that f satisfies the requirements. By assumption G is F_σ, hence $G \cap ([0,1) \setminus \mathbb{D})$ is F_σ in $[0,1) \setminus \mathbb{D}$,

so $B \cap ([0,1) \setminus \mathbb{D})$ is also F_σ in $[0,1) \setminus \mathbb{D}$. Then $B \cap [0,1)$ is $G_{\delta\sigma}$, which is a contradiction, as by [9, Ex 23.7] it is not $G_{\delta\sigma}$. $\square$

Proof. (Theorem 2.7) As we already mentioned, $BE(1,0)$ is clearly Borel and closed under addition, hence Lemma 2.9 shows that Theorem 2.8 applies. $\square$

We also formulate the following immediate corollary to Theorem 2.8, which is interesting in its own right.

Theorem 2.10 *If $G \subset \mathbb{R}$ is a Borel but not F_σ additive subgroup then G is immeasurable.* $\square$

Remark 2.11 In fact this theorem holds in every Polish group, and F_σ can be weakened to K_σ (that is, countable union of compact sets). In case the group is not abelian we have to use left-translations everywhere (including the definition of immeasurable.)

3 Immeasurable sets with arbitrary Borel class, Hausdorff or packing dimension

A nonempty open set is of positive Lebesgue measure, hence cannot be immeasurable. Our next theorem shows that we can find immeasurable sets in all other Borel classes.

For $1 \leq \alpha < \omega_1$ denote Σ^0_α, Π^0_α and Δ^0_α the additive, multiplicative and ambiguous Borel classes, respectively (see e.g. [9]). We say that a Borel set B is of exact class Σ^0_α (Π^0_α) if $B \in \Sigma^0_\alpha \setminus \Pi^0_\alpha$ ($B \in \Pi^0_\alpha \setminus \Sigma^0_\alpha$). We say that B is of exact class Δ^0_α if $B \in \Delta^0_\alpha$ but $B \notin \Sigma^0_\beta \cup \Pi^0_\beta$ for $\beta < \alpha$.

Theorem 3.1 *For every $1 \leq \alpha < \omega_1$ there exist immeasurable sets of exact Borel class Π^0_α. There exist immeasurable sets of exact Borel class Σ^0_α and Δ^0_α iff $2 \leq \alpha < \omega_1$.*

Proof. Δ^0_1: Impossible, as open sets are not immeasurable.

Σ^0_1: Impossible, for the same reason.

Π^0_1: Davies [5] constructed an immeasurable Cantor set D.

Δ^0_2: D minus a point is immeasurable by Lemma 2.2 *(ii)*.

Σ^0_2: $D + \mathbb{Q}$ is immeasurable by Lemma 2.2 (i). To see that its exact class is F_σ, first note that as D is immeasurable, it is of Lebesgue measure zero. Moreover it is closed, hence nowhere dense, therefore $D + \mathbb{Q}$ is a dense meager F_σ set. Thus it cannot be G_δ, as dense G_δ sets are comeager.

Π^0_2: The set of Liouville numbers is immeasurable, see Theorem 1.2. Larman's example [10] is also G_δ. Both examples are easily seen to be non-F_σ using the fact that nonmeager F_σ sets have interior points.

All other classes: These examples can be chosen to be groups: in [12] Borel groups of exact class Π^0_α, Σ^0_α and Δ^0_α are constructed (for $3 \leq \alpha < \omega_1$), and these are all immeasurable by Theorem 2.10. $\square$

Our next theorem shows that there are lots of immeasurable sets from the viewpoint of Hausdorff dimension as well.

Theorem 3.2 *For every $0 \leq \alpha \leq 1$ there exists an immeasurable subset of $\mathbb{R}$ of Hausdorff dimension α. In particular, these sets can be chosen to be additive subgroups of $\mathbb{R}$.*

Proof. First we consider the case $\alpha \neq 1$.

In [7] groups $G_\alpha \subset \mathbb{R}$ of Hausdorff dimension α are constructed for every $0 < \alpha < 1$. In fact, the proof also yields a nontrivial group G_0, but for our purposes it is sufficient to put $G_0 = \{0\}$. We will also use the fact that $G_\alpha \subset G_\beta$ for $\alpha < \beta$.

Unfortunately G_α is F_σ. Our goal is to find a group H 'sufficiently orthogonal to G_α' such that $G'_\alpha = G_\alpha + H$ is a non-F_σ group of dimension α.

The following lemma is probably well-known, however, we were unable to find suitable references for its second half, so we include a proof here.

Lemma 3.3 *Let M be a meager subset of $\mathbb{R} \setminus \{0\}$. Then the typical compact subset C of $\mathbb{R}$ (that is, comeager many elements of the space of compact subset of $\mathbb{R}$ endowed with the Hausdorff metric) possesses the following properties.*

(i) C is a Cantor set which is linearly independent over the rationals.

(ii) The group generated by C is disjoint from M.

Proof. *(i)* This is essentially [9, Ex 19.2] which follows from [9, Thm 19.1].

(ii) The complement of M contains a set that can be written as the intersection of countably many dense open sets. Hence it is enough to prove that for a fixed dense open set U, the group generated by C is in U.

For $(n_1, \ldots, n_k, n'_1, \ldots, n'_l) \in (\mathbb{N} \setminus \{0\})^{k+l}$ define

$$C(n_1, \ldots, n_k; n'_1, \ldots, n'_l) =$$

$$= \{n_1 c_1 + \ldots + n_k c_k - n'_1 c'_1 - \ldots - n_l c'_l : c_i, c'_j \in C \text{ are all distinct}\}.$$

Set $C(\emptyset) = \{0\}$. The union of all these sets as $(n_1, \ldots, n_k, n'_1, \ldots, n'_l)$ ranges over the countable sets $(\mathbb{N} \setminus \{0\})^{k+l}$ is the group generated by C, hence it suffices to prove that

$$C(n_1, \ldots, n_k; n'_1, \ldots, n'_l) \subset U$$

for the typical C. Define the map $f : \mathbb{R}^{k+l} \to \mathbb{R}$ by

$$f(x_1, \ldots, x_k, x'_1, \ldots, x'_l) = n_1 x_1 + \ldots + n_k x_k - n'_1 x'_1 - \ldots - n_l x'_l.$$

This map is clearly continuous and open, therefore $f^{-1}(U)$ is dense and open in $\mathbb{R}^{k+l}$. Then [9, Thm 19.1] states precisely that for the typical C the set

$$(C)^{k+l} = \{(c_1, \ldots, c_k; c'_1, \ldots, c'_l) : c_i, c'_j \in C \text{ are all distinct}\}$$

is contained in $f^{-1}(U)$. Therefore $C(n_1,\dots,n_k;n'_1,\dots,n'_l)\subset U$, which finishes the proof of the lemma. $\square$

The last easy statement we need before we define our group H is that every Cantor subset K of $\mathbb{R}$ contains a Cantor set K_0 of packing dimension zero. We need this additional step as the typical compact set has packing dimension 1. See [11] for the definition. Packing dimension (which is the same as upper packing dimension) of a set B will be denoted by $\dim_p B$, upper Minkowski dimension of B is denoted by $\overline{\dim}_M B$. We will use the well-known inequality $\dim_p B\le\overline{\dim}_M B$.

In order to prove this statement, we perform a Cantor-type construction as follows. We can clearly assume $K\subset[0,1]$. For every integer $m\ge 1$ divide $[0,1]$ into 2^m subintervals of length $\frac{1}{2^m}$. One can check that it is possible to choose recursively for every m a nonempty subfamily $\mathcal{I}_m$ of these intervals such that

- $\cup\mathcal{I}_m\supset\cup\mathcal{I}_n$ for $m<n$,
- $\mathrm{card}(\mathcal{I}_m)\le m$
- $\forall I\in\mathcal{I}_m\ \exists n\in\mathbb{N}$ and $J,J'\in\mathcal{I}_n, J\ne J'$, such that $J,J'\subset I$,
- $\forall I\in\mathcal{I}_m\ \mathrm{int}(I)\cap K\ne\emptyset$.

Define $K_0=\bigcap_{m=1}^\infty\cup\mathcal{I}_m$. This set is easily seen to be a Cantor set, and using the "box-counting" version of the upper Minkowski dimension (see [11, page 78]) yield $\overline{\dim}_M K_0=0$. Hence $\dim_p K_0=0$.

Now we complete the proof of Theorem 3.2. The only nonmeager Borel subgroup of $\mathbb{R}$ is $\mathbb{R}$ itself (since by [14, page 93] if B is a nonmeager Borel set than $B-B$ contains an interval). Therefore the sets $G_{1-\frac{1}{n}}\setminus\{0\}$ are meager. Apply Lemma 3.3 to $M=\cup_{n\in\mathbb{N}}(G_{1-\frac{1}{n}}\setminus\{0\})$ and obtain a compact set C. By the previous argument we can choose a Cantor set $C_0\subset C$ such that $\dim_p C_0=0$. Fix a Borel but not F_σ subset $B\subset C_0$, and define H as the group generated by B. We have to check that $G'_\alpha=G_\alpha+H$ is a Borel but not F_σ group of Hausdorff dimension α for every $0\le\alpha<1$.

As $G_\alpha\subset G_\beta$ for $\alpha<\beta$, we obtain $G_\alpha\cap H=\{0\}$ for every $0\le\alpha<1$.

First we show that H is Borel. As H is the group generated by B, it is the union of all the sets $B(n_1,\dots,n_k;n'_1,\dots,n'_l)$ (the notation was defined in the proof of Lemma 3.3). Therefore it is sufficient to show that these latter sets are all Borel. But since B is linearly independent, such a set is the continuous one-to-one image of the Borel set $(B)^{k+l}$ (as above, using the notation of the proof of Lemma 3.3), hence Borel.

Similarly, the natural map from the Borel set $G_\alpha\times H\subset\mathbb{R}^2$ onto $G'_\alpha=G_\alpha+H$ is continuous. Moreover, it is one-to-one, since $g_1+h_1=g_2+h_2$ implies $g_1-g_2=h_2-h_1$ and G_α and H are groups satisfying $G_\alpha\cap H=\{0\}$. Hence G'_α is Borel.

In order to show that G'_α is not F_σ will prove $G'_\alpha\cap C_0=B$, which is clearly sufficient as C_0 is compact and B is not F_σ. Suppose $g'=g+h\in C_0$, where $g\in G_\alpha, h\in H$. We want to show that $g'\in B$. Clearly, $g=g'-h\in C_0-H$.

But $C_0 - H$ is a subset of the group generated by C, and $G_\alpha \subset M$, so by Lemma 3.3 *(ii)* we obtain $g = 0$. Hence $h \in C_0$, and h is in the group generated by $B \subset C_0$. But C_0 is linearly independent, therefore $h \in B$ and we are done, as $h = g'$.

What remains to be shown is that $\dim_H G'_\alpha = \alpha$. Obviously $\alpha = \dim_H G_\alpha \leq \dim_H G'_\alpha$, so it suffices to prove $\dim_H G'_\alpha \leq \alpha$. It is well-known [11, Thm 8.10] that $\dim_p(X \times Y) \leq \dim_p X + \dim_p Y$, which implies $\dim_p(X_1 \times \ldots \times X_n) \leq \dim_p X_1 + \ldots + \dim_p X_n$. First we prove $\dim_p H = 0$. As H is the countable union of the sets $B(n_1, \ldots, n_k; n'_1, \ldots, n'_l)$, it is sufficient to show that $\dim_p B(n_1, \ldots, n_k; n'_1, \ldots, n'_l) = 0$. This set can be covered by a Lipschitz image of the set B^{k+l}, and Lipschitz images do not increase dimension, so it is enough to check $\dim_p B^{k+l} = 0$. But this clearly holds as $B \subset C_0$, so $\dim_p B \leq \dim_p C_0 = 0$, hence $\dim_p(B \times \ldots \times B) \leq \dim_p B + \ldots + \dim_p B = 0 + \ldots + 0 = 0$. So $\dim_p H = 0$.

Another part of [11, Thm 8.10] states that $\dim_H(X \times Y) \leq \dim_H X + \dim_p Y$. Choosing $X = G_\alpha$ and $Y = H$ yields $\dim_H(G_\alpha \times H) \leq \alpha + 0 = \alpha$. But G'_α is a Lipschitz image of $G_\alpha \times H$, hence $\dim_H G'_\alpha \leq \alpha$. This completes the proof for $\alpha < 1$.

Finally, we have to deal with the case $\alpha = 1$. As $G'_{1-\frac{1}{n}}$ is an increasing sequence of immeasurable groups of Hausdorff dimension $1 - \frac{1}{n}$, the union $\cup_{n=1}^\infty G_{1-\frac{1}{n}}$ is a group of Hausdorff dimension 1 which is immeasurable by Lemma 2.2 (i). $\square$

Remark 3.4 In [7] only $\dim_H G_\alpha = \alpha$ is proved, but $\dim_p G_\alpha = \alpha$ also holds (see sketch of proof below). Therefore one can easily check that a minor modification of the above proof yields $\dim_p G'_\alpha = \alpha$, hence Theorem 3.2 is also valid for packing instead of Hausdorff dimension.

Now we sketch the proof of $\dim_p G_\alpha = \alpha$. Since $\alpha = \dim_H G_\alpha \leq \dim_p G_\alpha$, it is enough to prove $\dim_p G_\alpha \leq \alpha$. Write $G_\alpha = \cup_{k_0 \in \mathbb{N}} \cup_{\kappa \in \mathbb{N}} \cup_{n \in \mathbb{Z}} (G_\alpha(k_0, \kappa) + n)$, where $G_\alpha(k_0, \kappa)$ is the set of points in $[0, 1]$ satisfying the requirement in the definition of G_α for fixed k_0 and κ (see the construction in [7]). It is sufficient to show $\dim_p G_\alpha(k_0, \kappa) \leq \alpha$ for fixed k_0 and κ. This set is a Cantor set, and it is not hard to see that the 'natural measure' on it (we always split the measure into equal parts in the Cantor-type construction) satisfies all conditions of [11, Theorem 6.11 (2)] with $A = G_\alpha(k_0, \kappa)$, $\lambda = 1$ and $s = \alpha + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Hence $\dim_p G_\alpha(k_0, \kappa) \leq \alpha$.

4 Concluding remarks

There are numerous questions arising naturally concerning immeasurability. Is it really weaker to require some translation invariant Borel measure than a Hausdorff or packing measure? The answer to the second version turns out to be positive, as Peres [15] showed that some of the so called Bedford-McMullen carpets are of zero or non-σ-finite packing measure for every gauge function,

while by recent results of the authors [6] none of these carpets are immeasurable. We do not know the answer to the other version: Does the existence of a translation invariant Borel measure that is positive and σ-finite on a Borel set B imply the existence of a Hausdorff measure with a suitable gauge function that is positive and σ-finite on B?

We may be overlooking something simple, but the following questions also seem to be open. We say that a Borel set B is *measured* if it is not immeasurable; that is, there is a translation invariant Borel measure for which B is positive σ-finite. Is the union of two measured sets also measured? (It can be shown using the construction in [5] that this is false for countably many sets.) Is the set of Liouville numbers a finite/countable union of measured sets? Is every Borel set a finite/countable union of measured sets?

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